

Sphere of Origin

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1 Defining the sphere of origin

1.1 The James Construction

[Whi78, VII §2, XII §1]. Let \mathcal{K} be Steenrods convenient category. Then there is an adjunction

$$\Sigma : \mathcal{K} \dashv \mathcal{K} : \Omega$$

One can ask what is the homotopy type of $\Omega\Sigma(X)$ for some space X and how it is related to the homotopy type of X , in some sense, how far is this adjunction from a homotopy equivalence. Because we know that looping shifts homotopy groups up and *in the stable range* suspending shifts them down this is a little bit like asking about the non-stable behaviour of the suspension. Given that looping produces a h-space structure via concatenation of loops it is reasonable that the answer might be sought in such “groupy” constructions. The answer to this question is indeed the so called “James construction”, which gives a very concrete model for this homotopy type.

The idea of the James construction is that it is the “free monoid on the points of X ”, the point set construction is as follows. Let X be based with base point e , then consider the finite sequences in X modulo the equivalence relation generated by the addition or removal of a finite number of the base point, this set we denote $J(X)$. For example

$$(a, e, b) \sim (a, b) \sim (a, b, e, e)$$

Clearly the base point is equivalent to the empty sequence $(e) \sim \emptyset$. This set is given a monoid structure by concatenating the sequences. Moreover there is a clear inclusion

$$X \rightarrow J(X), \quad x \mapsto (x)$$

Whitehead shows that this construction is universal as a monoid (any map from X to a monoid factors through $J(X)$). This monoid is topologised by defining the subsets $J^m(X)$ which are the elements up to equivalence that are of length m , noticing that $J(X) = \cup_m J^m(X)$ and then requiring that the inclusion

$$X^m \rightarrow J^m(X), \quad x_1, \dots, x_m \mapsto (x_1, \dots, x_m)$$

is continuous, giving the codomain the coarsest such topology. This construction is also functorial. It is then shown that

$$J(X) \simeq \Omega\Sigma(X)$$

1.2 The EHP sequence

There is a (homotopy) commutative diagram

$$\begin{array}{ccc}
 & & J(X) \\
 & \nearrow \text{inclusion} & \uparrow \\
 X & & \simeq \\
 & \searrow \text{adjoint of } \text{id}_{\Sigma X} & \downarrow \\
 & & \Omega\Sigma(X)
 \end{array}$$

This implies that there is an isomorphism

$$\pi_i(J(X), X) \cong \pi_i(\Omega\Sigma X, X),$$

i.e. that our homotopy equivalence behaves well under these pairs. One can therefore look at the long exact sequence of the pair on the left, and see that we have a commuting diagram

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & \pi_i(X) & \longrightarrow & \pi_i J(X) & \longrightarrow & \pi_i(J(X), X) \longrightarrow \cdots \\
 & & \searrow & & \parallel & & \parallel \\
 & & & & \pi_i(\Omega\Sigma X) & & \pi_i(\Omega\Sigma X, X) \\
 & & \searrow \Sigma & & \uparrow \cong, \text{ adjoint} & & \\
 & & & & \pi_{i+1}(\Sigma X) & &
 \end{array}$$

This makes precise the claim that we are investigating the non-isomorphism behaviour of the suspension. White head gives a further investigation into the homotopy group $\pi_i(J(X), X)$. For an $n - 1$ connected space X and $n \geq 2$ then there is a map

$$\pi_i(J(X), X) \rightarrow \pi_i J(X \wedge X)$$

which is iso for $i \leq 3n - 2$ and epi for $i = 3n - 1$. It is classical that $\pi_i \Omega\Sigma(X \wedge X) = \pi_{i+1} \Sigma(X \wedge X) = \pi_{i+1}(X * X)$, the join. Using these isomorphisms we see that in a range we get the exact sequence

$$\pi_{3n-2}(X) \xrightarrow{\Sigma} \pi_{3n-1}(\Sigma X) \xrightarrow{H} \pi_{3n-1}(X * X) \xrightarrow{\partial} \cdots$$

The Σ map is referred to as the E map from the German for suspension, and the boundary map is called P for some reason. Hence this is the EHP sequence.

1.3 The Sphere

For the sphere this sequence becomes

$$\pi_{3n-2}(S^n) \xrightarrow{\Sigma} \pi_{3n-1}(S^{n+1}) \xrightarrow{H} \pi_{3n-1}(S^{2n+1}) \xrightarrow{\partial} \cdots$$

and we are able to say a bit more about the maps. Actually the maps are understood for more than the sphere, but they are not covered in Whiteheads book (merely hinted at).

1.3.1 H

We lose nothing by being general here. Let W, X be pointed spaces and suppose we are given a map

$$f : (J^m(W), J^{m-1}(W)) \rightarrow (X, e)$$

there Whitehead constructs an extension

$$g : J(W) \rightarrow J(X).$$

To describe H we will apply this construction to the projection $X^{\times n} \rightarrow X^{\wedge n}$, which induces a map of pairs

$$(J^m(X), J^{m-1}(X)) \rightarrow (X^{\wedge n}, e)$$

using the construction we get an extension $g_n : J(X) \rightarrow J(X^{\wedge n})$. These codomains can be collected and then included into $X^{\wedge n} \xrightarrow{\iota} \vee_n X^{\wedge n}$, and using g_n we can define a map

$$G : J(X) \rightarrow J(\vee_n X^{\wedge n})$$

$$x \in J^m(X) \mapsto \prod_{n=1}^m J(\iota)(g_n(x))$$

One has to check that this is well defined. There is again a projection $\vee_n X^{\wedge n} \xrightarrow{p_m} X^{\wedge m}$. The result of all this is a diagram

$$\begin{array}{ccc} \pi_* J(X) & \xrightarrow{\pi_*(J(p_m) \circ G)} & \pi_* J(X^{\wedge n}) \\ \parallel & & \parallel \\ \pi_* \Omega \Sigma(X) & & \pi_* \Omega \Sigma(X^{\wedge n}) \\ \parallel & & \parallel \\ \pi_{*+1} \Sigma X & \xrightarrow{j_n} & \pi_{*+1} \Sigma(X^{\wedge n}) \end{array}$$

defining the Hopf-James invariants j_n . To summarise what is going on, we include into the smashes, then include into the wedge product, we then multiply the different representatives and finally project again. The index of the invariant is the degree of the smash that we include into, not the degree of the projection. Then the claim, that one has to prove, is that $H = j_2$.

Example...

1.3.2 P

First note that the double suspension for spheres is an isomorphism. **Check.** Looking at the LES we have the following

$$\begin{array}{ccccccc} \pi_i(S^n) & \xrightarrow{\Sigma} & \pi_{i+1}(S^{n+1}) & \xrightarrow{H} & \pi_{i+1}(S^{2n+1}) & \xrightarrow{\partial} & \pi_{i-1}(S^n) \\ & & & & \uparrow & \nearrow \xi & \\ & & & & \cong, \Sigma^2 & & \\ & & & & \pi_{i-1}(S^{2n-1}) & & \end{array}$$

Whitehead tells us that $\xi(\alpha) = [\text{id}_{S^n}, \text{id}_{S^n}] \circ \alpha$ where the square brackets denote a (different) Whitehead product.

2 Comparing to Disc of origin

Recall that there is a map $\Theta_n \rightarrow \text{coker}(J_n) = \pi_n^S$. Then we have the following diagram

$$\begin{array}{ccc}
 \pi_{n-1} \text{Diff}_{\partial}(D^0) & & \pi_n(S^0) \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 \pi_2 \text{Diff}_{\partial}(D^{n-3}) & & \pi_{2n} S^n \\
 \downarrow & & \downarrow \Sigma \\
 \pi_1 \text{Diff}_{\partial}(D^{n-2}) \longrightarrow \pi_1 \mathcal{C}(D^{n-3}) & & \pi_{2n+1} S^{n+1} \xrightarrow{H} \pi_{2n+1} S^{2n+1} \\
 \downarrow & & \downarrow \\
 \pi_0 \text{Diff}_{\partial}(D^{n-1}) & & \pi_{2n+2} S^{n+2} \\
 \downarrow \sim & & \downarrow \sim \\
 \Theta_n & \xrightarrow{\text{“quotient”}} & \text{coker}(J_n) \xleftarrow{\text{quotient}} \pi_n^S
 \end{array}$$

which produces two filtrations of the cokernel of the J homomorphism. Some remarks, the verticle maps on the left are degree $(-1, 1)$ while on the right they are $(1, 1)$. The image of both of these maps is controlled by the kernel of some other “obstruction”, on the left the map to the concordance space and on the right the H map. We want to compare how pulling back in one sequence compares to pulling back in the other. For a class $\zeta \in \text{coker}(J_n)$ we define the disc and sphere of origin as

$$\text{DOO}(\zeta) := \max\{m : [\Sigma] = \zeta \text{ and } \Sigma \in \text{Im}(\pi_* \text{Diff}_{\partial}(D^m) \rightarrow \Theta_n)\}$$

$$\text{SOO}(\zeta) := \min\{m : [\gamma] = \zeta \text{ and } \gamma \in \Sigma^{n+2-m} \pi_* S^m\}$$

A starting point is given by considering the derivate of the diffeomorphisms, this defines a map

$$d : \text{Diff}_{\partial}(D^n) \rightarrow \text{Hom}((D^n, \partial D^n), (\text{SO}_n, \text{id})) \simeq \Omega^n \text{SO}_n$$

$$f \mapsto (x \mapsto D_x f)$$

this map is rationally trivial (on homotopy groups) and integrally non-trivial [CSS23]. If we look at the map on homotopy groups then we can define the “framed disc of origin” by the exact sequence

$$0 \rightarrow \pi_i \text{Diff}_{\partial}^{fr}(D^n) := \ker \pi_i(d) \rightarrow \pi_i \text{Diff}_{\partial}(D^n) \xrightarrow{\pi_* d} \pi_i \Omega^n \text{SO}_n \rightarrow 0$$

Recall a framing of a submanifold is a basis of the normal bundle at each point, that is a non-vanishing section or a “trivialisation”, by [CSS23, Lemma 2.1] the derivative map classifies normal bundle structures, and hence being in the kernel is having a trivial normal bundle, or a framing. Let $\xi \in \pi_{n-k} \text{Diff}_{\partial}^{fr}(D^k)$ correspond to Σ^{n+1} a homotopy sphere. What the authors show is that the normal bundle of some explicit embedding $\Sigma \hookrightarrow S^{n+k+1}$ is obtained by clutching with $d\xi$ and hence the derivative defines a one-one correspondence between $\{\text{normal bundles of embeddings } \Sigma \rightarrow \mathbb{R}^{n+k+1}\} \leftrightarrow \pi_n \text{SO}_k$ (possibly modulo ± 1 on the right). **Return to proof.**

Lemma. $\text{SOO} \geq \text{framed DOO}$.

Proof. Let $\Sigma \in \Theta_{n+1}$ pull back to an element $\xi \in \pi_{n-k} \text{Diff}_0^{fr}(D^k)$. Then as above we see that there is an embedding $\Sigma \hookrightarrow S^{n+k+1}$. The normal bundle of this embedding is trivialisable because by definition the derivative map on homotopy groups maps it to zero (and the correspondence above). This trivialisisation defines a framing as discussed and so we can apply the Pontryagin-Thom coorespondence from framed submanifolds of S^{n+k+1} to homotopy classes of maps $S^{n+k+1} \rightarrow S^k$, as Σ^{n+1} is codimension k . Thus we obtain a class $\zeta \in \pi_{n+k+1} S^k$, checking the stable range we have that $n+k+1 \leq 2k$ only when $n \leq k-1$ and so this class may or may not be stable.

What we want to show is that the stabilisation of ζ is Σ in the coker J_n . **Non-trivial, Diarmuid claims that the relevant commutative diagrams should be somewhere in one of his papers with Shick and or Stiemele.**

Thus we have shown that if you have disc of origin at most k then the sphere of origin is at most k , and in particular the sphere of origin is less than or equal to the disc of origin.

Lemma (Unsubstantiated Conjecture). *They are equal.*

Proof. (Sketch) Let $x \in \pi_k^S$ and suppose $SOO(x) \leq d$. Then there is some map $f : S^{k+d} \rightarrow S^d$ that suspends to x .

Look at the fiber over the base point of this map $V^k = f^{-1}(*)$. We can ask when this fiber is a homotopy sphere. ‘‘Ambient surgery’’ aparently gives sufficient conditions to resolve this problem, likely true stably.

So if we assume that $V^k = \Sigma^k$. Then the inclusion $V^k = \Sigma \hookrightarrow S^{k+d}$ **somehow this gives an element** $\tilde{x} \in \pi_k \widetilde{PL}_d/O_d$ then the zig zag

$$\pi_k \widetilde{PL}_d/O_d \rightarrow \widetilde{PL}/O \xleftarrow{\cong} \pi_k PL/O \rightarrow \pi_k G/O$$

Finally something about

$$\pi_k PL_d/O_d \rightarrow \pi_k \widetilde{PL}_d/O_d \rightarrow \pi_k \widetilde{PL}_d/PL_d??$$

‘‘Compare them at odd primes look in Burghaelea Lashoff II section 7’’

References

- [CSS23] Diarmuid Crowley, Thomas Schick, and Wolfgang Steimle. The derivative map for diffeomorphism of disks: an example. *Geometry & Topology*, 27(9):3699–3713, December 2023.
- [Whi78] George W. Whitehead. *Elements of Homotopy Theory*, volume 61 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 1978.